ISSN 2229-5518

Image Reconstruction from Finite Projections with Geometry Transfer

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Abstract— This paper proposes a method of image reconstruction using the concept of geometry transfer of the image with discrete paired transform combined with radon transform. It is based on the fact that the integral geometry of the image can be transferred from the image plane to Cartesian lattice. This is achieved by converting the line integrals obtained from the image and converting them to line sums of the corresponding discrete image. Radon transform is combined with paired transform to obtain exact reconstruction with finite number of images. This is an effective and improved method. The number of calculations required is reduced as the number of projections is limited and the reconstruction is exact. This method provides exact reconstruction even when the projections are noisy.

Index Terms—analytical reconstruction, discrete paired transform, geometry transfer, radon transform,tomographic imaging .

1 Introduction

Image reconstruction has numerous applications in the real world, most of which are in the field of medical image processing. The reconstruction of images during various scans like CT, PET etc. is the most useful implementation domain of the reconstruction process. Other applications might be graphics designing, multimedia applications etc.

There are many existing methods of image reconstruction that are either iterative or analytical. The proposed method is an analytical method in which the main concepts involved are: geometry transfer of images from the integral geometry to Cartesian lattice[1]; the use of discrete paired transform that reduces the redundancy of projections and provides partial reconstruction[1][2]; Radon transform[3]. Radon transform usually requires infinite number of projections for exact image reconstruction, but with the combination of geometry transfer and paired transform exact reconstruction is possible even though the number of projections is limited.

The proposed method is an efficient method for image reconstruction. The geometry transfer accounts for easy calculations. Paired transform removes similar projection and hence eliminate redundancy. Radon transform provides the radon projection. The inverse of these transforms provides the reconstructed image. The inverse transforms are averaged and combined to provide the reconstructed image.

The remainder of this paper is organized as follows: Section II lays down the existing methods, Section III describes the proposed method, Section IV has the Results of different experiments, Section V gives the conclusion followed by the acknowledgement and references.

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2 Existing Work

There has been substantial research in the field of image reconstruction and different methods have been proposed at different times. The image reconstruction methods can be mainly categorized into two types: iterative method; and analytical method[8][9]. The analytical reconstruction approaches, in general, try to formulate the solution in a closed-form equation. Iterative reconstruction tries to formulate the final result as the solution either to a set of equations or the solution of an optimization problem, which is solved in an iterative fashion. Analytical reconstruction is considered computationally more efficient while iterative reconstruction can improve image quality. It should be pointed out that many reconstruction algorithms do not fall into these two categories in the strict sense. These algorithms can be generally classified as hybrid algorithms that leverage advanced signal-processing, image-processing, and analytical reconstruction approaches.

The major methods available that provide exact reconstruction either require infinite number of projection for the reconstruction; or involve very complex calculation performed multiple times iteratively. The methods like Fourier slice theorem, inverse radon transform, and filtered back projection are the existing methods. Radon transform on its own requires infinite number of projections to reconstruct the image exactly. Often infinite projections are unavailable.

3 Proposed Method

The proposed method of reconstruction tries to overcome the shortcomings of the existing methods and introduces a unique combination of technologies for exact reconstruction with limited data, which the real world scenario most of the time. The method is based on a simple fact that geometry of an image can be transferred from the continuous geometry of the plane image to Cartesian lattice of the corresponding discrete image[1]. This geometry transfer enables us to view the image as a discrete image with NxN image elements.

ISSN 2229-5518

3.1 Geometry Transfer

The continuous image f(x,y) is converted to discrete image. It is represented as digital image containing a finite number of cells called image elements(IE) as given in [1]. It is assumed that the original image f (x, y) occupies the square region [0, 1]×[0, 1] which is divided into N2 image elements by the N × N Cartesian lattice, where N > 1. The image elements are numbered as I En,m, where n,m = 0 : (N−1), and the image is considered in matrix form fn,m. To transfer geometry two co-ordinate systems are used to represent the same image as shown in Figure 1.

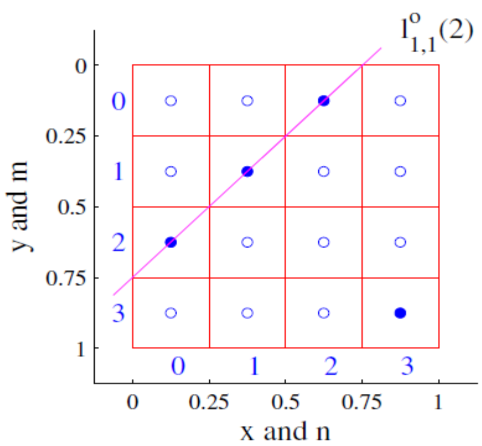

Fig. 1. Two co-ordinate system for image: represented in continuous (x-y co-ordinate system) as well as discrete (Cartesian lattice) co-ordinate system, and line l°1,1(2).

The first co-ordinate system (x, y) is for the image f (x, y) on the square [0, 1] × [0, 1]. The second co-ordinate system (n,m), where n and m are integers, is for the lattice XN,N located in the square which is used for the discrete image fn,m. Parameters x and n run from left to right, and parameters y and m run from top to bottom. This is shown in Figure 1 for the case N = 4. The first point of the discrete image, f0,0, is in the point with coordinates (x, y) = (1/8, 1/8).

XN,N is the Cartesian lattice {(p, s); p, s = 0, 1, . . . , (N − 1)}. Given the frequency-point (p, s) ∈ XN,N , such that g.c.d.(p, s) = 1, we consider the lines l°(t)=l°p,s (t)={(n,m); pn+sm = t}, t =0 : (p+s)(N−1),

on the square lattice XN,N for projections. These lines are the arithmetic rays. For example, the ray l°1,1(2) for N = 4 is shown in Figure 1. The same lines can be shown on the square [0, 1] × [0, 1] by the equation l(t) = l p,s(t) = {(x, y); px + sy = t/N + (p + s)/(2N)}, t = 0 : (p + s)(N − 1). They are called the geometrical rays in this case to distinguish the discrete and continuous cases. The two types of rays denoted by l°(t) and l(t) respectively, consider the same set of t, t = 0 : (p + s)(N − 1). The generator (p, s) defines the slope, ϕ(p, s) = π − tan−1(p/s), of these rays. The set of line-integrals {wl(t); l(t) = l p,s(t), t = 0 : (p + s)(N − 1)} is called the (p, s)-projection of the image. The angle of this projection is ϕ(p, s) − π/2. Thus for geometry transfer the arithmetic rays from the Cartesian lattice corresponding the geometric rays of the projection may be considered.

3.2 Geometry Transfer

The tensor representation of the discrete image is obtained by the following formula,

$$F_{kp \bmod N, ks \bmod N} = \sum_{t=0}^{N-1} f_{p,s,t} W^{kt}, \quad k = 0 : (N-1) \qquad (1)$$

where W = WN = exp(−2π j/N). fp,s,t is component of the splitting signal in the image element (p,s) which is explained in [1] and [2]. Given (p,s) the components of the splitting-signal are the sums of the image fn,m along the parallel lines on the lattice

$$f_{p,s,t} = \sum_{(n,m)\in X} \{f_{n,m};\ np + ms = t \bmod N\}$$

The set JN,N of frequency-points (p, s), or generators, of the splitting-signals is selected in a way that covers the Cartesian lattice XN,N with a minimum number of subsets Tp,s. The set JN,N contains 3N/2 generators and can be defined as JN,N ={(1,s); s=0 : (N−1)}∪{(2 p,1); p=0 : (N/2−1)}

The tensor representation of the image is unique and the image can be represented through equation (1) which is the 2D DFT of the image.

The tensor transform is redundant; there are many intersections at some frequency points. Paired transform is used to represent the image at a unique set of splitting signals. The splitting-signals in paired representation carry the spectral information about the 2D DFT at N/2k+1 frequency points. The paired transform can be represented as follows,

$$F_{(2m+1)p \bmod N,(2m+1)s \bmod N} = \sum_{t=0}^{L-1} [f'_{p,s,2^k t} W_{2L}^{t}] W_L^{mt} \qquad (2)$$

where L = N/2k+1 and m = 0 : (L − 1).The paired transform thus defines projection data at unique points in the image.

3.3 Radon Transform

The 2D Radon transformation is the projection of the image intensity along a radial line oriented at a specific angle. Radon expresses the fact that reconstructing an image, using projections obtained by rotational scanning is feasible. The value of a 2-D function at an arbitrary point is uniquely obtained by the integrals along the lines of all directions passing the point. The Radon transformation shows the relationship between the 2-D object and its projections[3].

Suppose a 2D function f(x,y) as shown in figure 2. Integrating along the line, whose normal vector is in direction θ, results in the function g(s, θ) which is the projection of the 2D image f(x,y) on the axis s of direction θ. When s is 0, the function g has value g(0, θ) which can be obtained by the integration along the line that passes through the origin of (x,y)-coordinate.

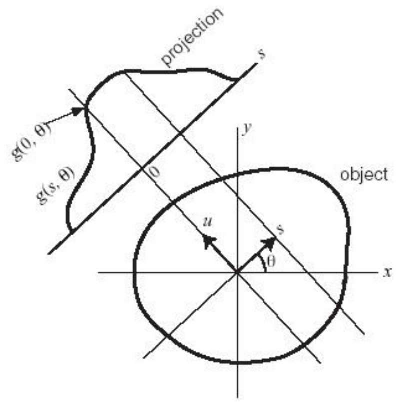

Fig. 2. Radon Transform computation

The points on the line whose normal vector is in θ direction and passes the origin of (x, y)-coordinate satisfy the equation:

$$\frac{y}{x} = \tan(\theta + \frac{\pi}{2}) = \frac{-\cos\theta}{\sin\theta} \Rightarrow$$

$$\Rightarrow x\cos\theta + y\sin\theta = 0$$

The Radon transform on an image f(x,y) on a set of angles computes the projection along those angles. The result is the sum of pixel intensities in these directions. Radon transform is obtained as follows

$$R(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(\rho - x\cos\theta - y\sin\theta)dx\, dy \quad (3)$$

Multiple parallel beam projections of the image from different angles are taken. The radon transform thus gives angular projection along a given set of angles.

3.4 Inverse Projections

The transforms give the projection data of the image. The reconstruction of the image is obtained by computing the inverse transform from the projection data obtained. In case of 2D discrete paired transform, the inverse transform is the sum of the image along the arithmetic rays. The inverse 2D DPT is obtained from the discrete image as follows

$$f_{n,m} = \frac{1}{2N}\sum_{k=0}^{r-1}\frac{1}{2^k}\sum_{(p,s)\in 2^k J_{2^{r-k},2^{r-k}}} f'_{p,s,(np+ms) \bmod N} + \frac{1}{N^2}f'_{0,0,0} \quad (4)$$

Filtered Back Projection is used to obtain the inverse Radon transform. In this method the approximation of the image is obtained based on the projections in the columns of the projection data R. As the number of projections increases the accuracy of the reconstructed image also increases. The inverse of Radon transform is calculated by the following equation,

$$f(x,y) = \int_{-\pi/2}^{\pi/2} \rho \cdot R_\theta(s(x,y))\, d\theta \quad (5)$$

The proposed method combines the inverse 2D DPT and inverse radon transform. The weighted average of the two inverse transforms is taken to combine the individual outputs. By combining the two methods the drawbacks of each method are removed. The radon transform originally requires infinite number of projections for exact and accurate reconstruction. But combining the discrete paired transform gives accurate reconstruction with fewer numbers of projections. The output obtained is an exact reconstruction even though the amount of data available is limited which is the real world scenario. The proposed method is efficient and accurate. The weakness of each method is overcome by the other. Geometry transfer and finite number of projections give efficient results even though it is not that accurate.

The original image is given as input to obtain projection from the image. Then reconstruction of the image is done using the projection data. The results of the proposed method are shown in the figure 3.

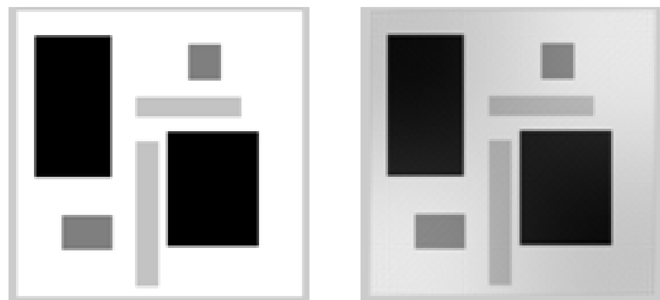

Fig. 3. The original image (a) and the reconstructed image (b) obtained from the projections.

The image obtained is an exact reconstruction of the original image. Reconstruction is done for grayscale images. The input image is converted to grayscale image and then projection taken. The images are shown in figure 4.

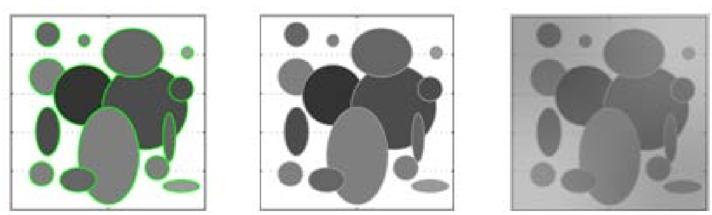

Fig. 4. The original color image (a) is coverted to grayscale image (b) which is then reconstructed (c).

4 Experiments and Results

The proposed method was tested with noisy image. The noise level was varied and the different images tested. The reconstruction was exact in all the cases. Exact reconstruction is obtained even with noisy projection data. This shows the robustness of the proposed method. Gaussian noise is introduced into the input image which is reconstructed. Figure 5 shows the resulting images from testing the method on Shepp-Logan phantom image.

The standard deviation is varied and then the image tested. Such an image gives noisy data as input for the generation of

ISSN 2229-5518

projections. In real world scenarios the projection data obtained can be noisy and unclear. This method provides exact and accurate reconstruction in all the cases. The method is efficient and advanced. The method of geometry transfer and

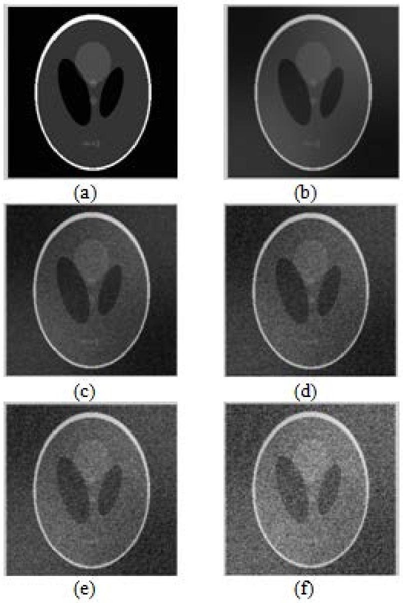

Fig. 5. (a) Shepp Logan image (b) the reconstructed image and the noisy images with (c) SD = 0.025 (d) SD = 0.5 (e) SD = 0.1 (f) SD = 0.2

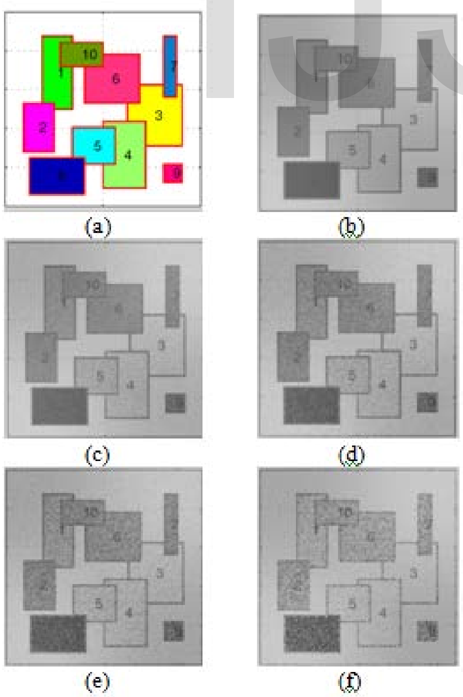

Fig. 6. The original image (a) , reconstructed image (b) and reconstruction of images with standard deviation (c) 0.025 (d) 0.05 (e) 0.1 (f) 0.2

paired transform accounts for the limited number of projections required or exact reconstruction. The radon transform and discrete paired transform together provide the exact reconstruction. Figure 6 shows multiple squares image which is reconstructed without noise and with varying levels of noise. The error calculation is shown in Table I. The high SNR and PSNR values show the efficiency and robustness of this method.

TABLE I. IMAGE RECONSTRUCTION SNR AND PSNR FOR FIGURE 6

Standard Deviation	SNR	PSNR
0.025	59.4513	61.1776
0.050	59.2927	60.8935
0.100	59.0275	60.2623
0.200	58.4285	59.0809

5 Conclusion

The proposed method is an efficient and accurate method of image reconstruction. The experiments were performed in MATLAB on intel dual core i5 processor. This method provides exact reconstruction of the image using geometry transfer and 2D DPT along with Radon transform; using a finite number of projections. Exact reconstruction is obtained even though the number of projections is limited. This is a robust method that reconstructs the image exactly and accurately even when the projection data is noisy.

Acknowledgments

I extend my heartfelt thanks to Mr. Vishnu and Mr. Neeraj who have been very helpful during the development of the method proposed in this paper. I'd also like to thank my teachers and family for their guidance and support.

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ISSN 2229-5518
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